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Irreversible quantum baker map

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We propose a generalization of the model of classical baker map on the torus, in which the images of two parts of the phase space do overlap. This transformation is irreversible and cannot be quantized by means of a unitary Floquet operator. A corresponding quantum system is constructed as a completely positive map acting in the space of density matrices. We investigate spectral properties of this superoperator and their link with the increase of the entropy of initially pure states.

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In past years a reexploration of a finite-dimensional Hilbert space $\mathcal{H}$ and the space of density operators acting on it took place. Emerging interest in properties of mixed quantum states is stimulated by research on the decoherence phenomena [1,2], and recent developments in modern applications of quantum mechanics including quantum information, cryptography, and computing [3]. The concept of mixed states is crucial while analyzing nonunitary quantum evolution, necessary to describe processes of measurement and interaction with an environment [4,5], or by looking for quantum analogs of classically irreversible dynamical systems. The latter issue was considered in the pioneering papers of Graham [6], who analyzed an irreversible map on the cylinder and found a corresponding quantum dynamics in an infinite Hilbert space.

Any quantum map $\mathbf{\Lambda}$ should send positive density operators into other positive operators. Moreover, since any system under consideration, described by a density operator, may be coupled to an environment, so $\mathbf{\Lambda}\otimes\mathbf{1_m}$ should be positive for any extension of $\mathbf{\Lambda}$ by the m-dimensional identity matrix $\mathbf{1_m}$. This property is called *complete positiveness* (CP) [7]. If the classical dynamics preserves probability, then the corresponding quantum map should preserve the trace of the density operator.

Research on quantum analogs of classically chaotic dynamical system often concentrates on two-dimensional area preserving maps. The most popular examples include the classical baker map and the Arnold cat map. They were quantized by finding the corresponding unitary operators, which act on a finite-dimensional Hilbert space $\mathcal{H}$ (see, e.g., Ref. [8], and references therein).

In this paper, we propose a generalization of the classical and quantum baker maps. The classical map proposed is irreversible, and therefore its quantum counterpart cannot be represented by a unitary operator. The classical map transforms a unit square into a rectangular subset of it, while the quantum map is a completely positive, trace preserving superoperator acting in the space of density matrices of a fixed size. Our research is related to recent papers of Soklakov and Schack [8], and Saraceno and Vallejos, who quantized a dissipative version of the baker map [9], and also studied a stochastic system devised to take into account the effects of decoherence [10]. However, the system analyzed here is different, since it is not dissipative, it conserves the probability and is deterministic. The quantization of the system on the torus leads to a map acting on finite dimensional Hilbert space, in contrast to the model discussed by Graham [6]. Therefore, the irreversible quantum baker map is suitable to investigate the spectral properties of the superoperator and its semiclassical regime. Furthermore, our approach allows one to introduce an irreversibility into any unitary quantum map on the torus. Hence, by analyzing different unitary quantum maps one may investigate the role of classical chaos in the speed of decoherence in the quantum system.

The standard baker map is a transformation of the unit square I, a model of a finite phase space, onto itself. It consists of stretching the square in one direction, labeled q, and squeezing it in another direction (labeled p) by the factor of 2. After the stretching procedure, the baker cuts the rectangle into two pieces and places the right piece at the top of the left one, as shown in Fig. 1 (transformation Θ). Assume that instead of doing this, the sloppy baker puts the right piece a bit too low, in such a way that a $\Delta/2$ overlap with the left piece occurs. This effect is described by the transformation L_Δ (formally an interval translation map acting in the p di-

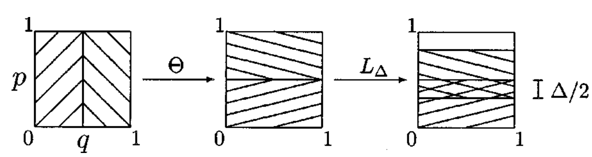

FIG. 1. Classical sloppy baker map; after the original baker transformation Θ, the top half of the square is shifted down by $\Delta/2$ (operator L_Δ).

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rection [11]), which shifts all points from the top half of the square ($p>1/2$) down by $\Delta/2$. The formal definition of the *classical sloppy baker map* is

$$\Theta_\Delta:\begin{pmatrix} q\\ p\end{pmatrix}\xrightarrow{B}\begin{pmatrix} 2q-[2q]\\ \frac{1}{2}(p+[2q])\end{pmatrix}\xrightarrow{L_\Delta}\begin{pmatrix} 2q-[2q]\\ \frac{1}{2}(p+[2q](1-\Delta))\end{pmatrix},\tag{1}$$

where $[x]$ denotes the integer part of x and the parameter Δ belongs to $[0,1]$. The map Θ_Δ is not reversible for $\Delta>0$, because any point for which $p\in([(1-\Delta)/2],1/2)$ has two preimages, while the points with $p>1-(\Delta/2)$ have none.

We will use density distribution f on the square, $\int_I f(\alpha)d\alpha=1,\ f\geqslant 0$, where α is a short notation for the pair (q,p). The map Θ_Δ generates the Frobenius-Perron operator acting in the space of classical density distributions,

$$\mathcal{M}f(\alpha)=\int_I f(\alpha')\delta(\alpha-\Theta(\alpha'))d\alpha'.\tag{2}$$

Since the map is not dissipative, and $\Theta_\Delta(I)\subset I$, the operator $\mathcal{M}$ preserves the probability, $\int_I\mathcal{M}f(\alpha)d\alpha=\int_I f(\alpha)d\alpha=1$. The density $f^*(\alpha)=1/(1-\Delta)$ for $\alpha\in[0,1]\times[0,1-\Delta)$ and 0 elsewhere is invariant under the action of the operator $\mathcal{M}f^*=f^*$. Several versions of quantum baker map on the torus are known [12–14]. We use the first form of the quantum operator proposed by Balazs and Voros [12],

$$\mathbf{B}=F_N^\dagger\begin{pmatrix} F_{N/2} & 0\\ 0 & F_{N/2}\end{pmatrix},\tag{3}$$

where F_N denotes the N point discrete Fourier transformation, $[F_N]_{kl}=(1/\sqrt{N})e^{-2\pi ikl/N},k,l=0,\ldots,N-1$. Since the sloppy map Θ_Δ does not enjoy the symmetry of the original baker map, we will not need the symmetric quantum model introduced by Saraceno [13]. Unitary operator $\mathbf{B}$ acts on the N-dimensional Hilbert space $\mathcal{H}_N$, where N is even. The classical map Θ_Δ is irreversible, so its quantization cannot be achieved by means of unitary operators. The quantum operator $\mathbf{\Lambda_\Delta}$ corresponding to the classical map L_Δ should act on the space of mixed quantum states, and may be realized by a superoperator. Any superoperator $\mathbf{\Lambda}$ that defines a completely positive map may be written in the so-called Kraus form [7],

$$\mathbf{\Lambda}(\rho)=\sum_{i=1}^{K}A_i\rho A_i^\dagger,\tag{4}$$

where ρ is a density matrix and K is finite. If operators A_i fulfill the condition

$$\sum_{i=1}^{K}A_i^\dagger A_i=\mathbf{1}_N,\tag{5}$$

the map $\mathbf{\Lambda}(\rho)$ is trace preserving. The classical map Θ_Δ transforms the bottom half of the square I into itself and shifts the top one down by $\Delta/2$. The two halves of I are transformed separately. Therefore we split the phase space into bottom and top, and introduce two projection operators D_b and D_t, which when written in the eigenbasis of position operator have the form

$$D_b=F_N^\dagger\begin{pmatrix}\mathbf{1}_{N/2} & 0\\ 0 & 0\end{pmatrix}F_N;\quad D_t=F_N^\dagger\begin{pmatrix}0 & 0\\ 0 & \mathbf{1}_{N/2}\end{pmatrix}F_N,\tag{6}$$

Notice that the superoperator $\mathbf{\Lambda_M}(\rho)=D_b\rho D_b^\dagger+D_t\rho D_t^\dagger$ corresponds to the up/down measurement process, and the Kraus operators $A_1=D_b$ and $A_2=D_t$ fulfill the condition (5). To construct a quantum shift transformation $\mathbf{\Lambda_\Delta}$, we will use the unitary operator of translation in momentum,

$$V|k\rangle=|k+1\rangle,\quad V^N=\mathbf{1}_N.\tag{7}$$

Here $|k\rangle$ denotes the discrete eigenstate of momentum which is periodic, $|k+N\rangle=|k\rangle$ [13]. For $k=1$, the state is localized at the bottom of I. Then the vertical shift of the top half of I by $\Delta/2$ is realized by the translation operator (7) acting on the previously measured system,

$$D_t'=V^{-N\Delta/2}D_t.\tag{8}$$

We assume here that the exponent is integer; however, this construction might be generalized for any real Δ. Since the position of the bottom part remains unchanged, the entire quantum transformation $\mathbf{\Lambda_\Delta}$ reads

$$\mathbf{\Lambda_\Delta}(\rho)=D_b\rho D_b^\dagger+D_t'\rho D_t'^\dagger.\tag{9}$$

This superoperator resets to the off-diagonal blocks of the ρ matrix zero in the p representation. This is related to the fact that to displace one-half of the torus we need to perform a measurement, which implies decoherence. Thus even for $\Delta=0$, the operator $\mathbf{\Lambda_\Delta}$ differs from identity, but the effect of the measurement becomes negligible in the classical limit $N\to\infty$.

Using the above superoperator ($\mathbf{\Lambda_\Delta}$), we construct the entire quantum sloppy baker map,

$$\mathbf{B_\Delta}(\rho)=\mathbf{\Lambda_\Delta}(B\rho B^\dagger)=D_bB\rho B^\dagger D_b^\dagger+D_t'B\rho B^\dagger D_t'^\dagger.\tag{10}$$

Note that the Kraus operators $A_1=D_bB$ and $A_2=D_t'B$ fulfill condition (5), with $K=2$.

To demonstrate that quantum system defined by Eq. (10) corresponds to the classical sloppy baker map we compare the classical and the quantum structures in the phase space. In order to define quantum quasiprobability distribution, we use a family of states localized at points of the square $N\times N$ lattice in the phase space constructed by means of translation operators [13]. The operator U of translation in position is defined similarly to V,

$$U|n\rangle=|n+1\rangle,\quad U^N=\mathbf{1}_N,\tag{11}$$

where $|n\rangle$ are position eigenstates, satisfying $|n+N\rangle=|n\rangle$. As a reference state we choose arbitrarily the wave packet $|\frac{1}{2},\frac{1}{2}\rangle$ localized in $(\frac{1}{2},\frac{1}{2})$,

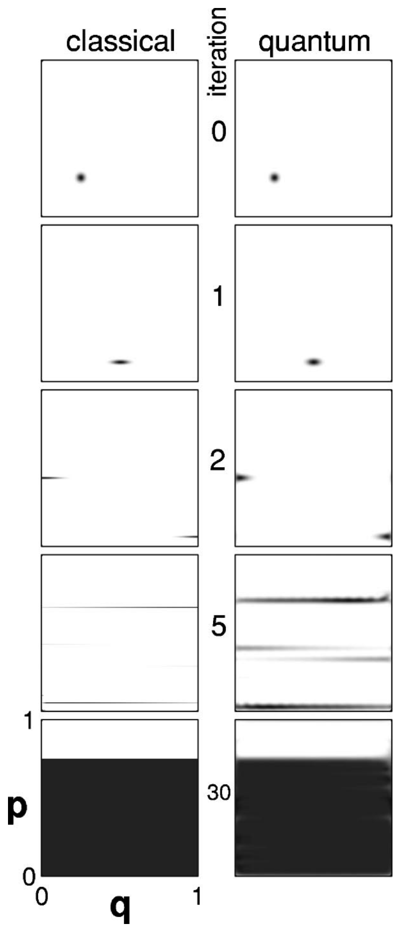

FIG. 2. Sloppy baker map with $\Delta=1/4$. Time evolution of an initially localized classical density concentrated at $\alpha_0=(0.25,0.25)$ (left) and Husimi representation (14) of the density matrix of an initially pure state localized in the same point α_0 and iterated by the quantum map (10) for $N=512$ (right).

$$\langle n|1/2,1/2\rangle=(2/N)^{-1/4}e^{-\{\pi[n-(N/2)]^2/N\}-i\pi n}, \qquad (12)$$

which becomes Gaussian for $N\to\infty$. We translate it to any point (q,p), where Nq and Np are integers (N is even),

$$|q,p\rangle=V^{Np-(N/2)}U^{Nq-(N/2)}|1/2,1/2\rangle. \qquad (13)$$

These states allow one to define a Husimi representation in the phase space of any mixed quantum state ρ,

$$H_\rho(q,p)=\langle q,p|\rho|q,p\rangle. \qquad (14)$$

We analyzed the evolution of an exemplary state $|\alpha_0\rangle$ localized at α_0 and the classical transformation of the corresponding density distribution. On the left-hand side of Fig. 2, we present the classical density and its image after $T=1,2,5$, and 30 iterations of the Frobenius-Perron operator (2). The right-hand side shows the Husimi representations (14) of the initially pure quantum state $|\alpha_0\rangle\langle\alpha_0|$ and its images after T actions of the superoperator $\mathbf{B}_\Delta$. The quantum quasiprobability distribution H_ρ is localized in the same region of the phase space as the classical density distribution. Since the Husimi distribution may resolve quantum phase-space structures at the length scale of the order of $\hbar^{1/2}\propto N^{-1/2}$, the classical density becomes narrower than its quantum counterpart already after first iteration.

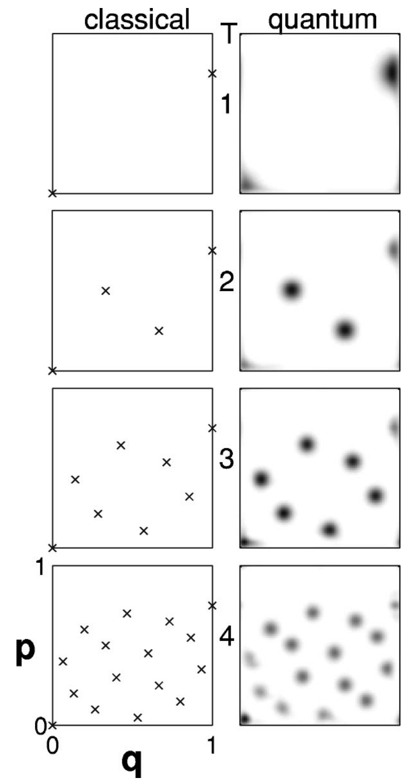

FIG. 3. Classical orbits of period $T=1,2,3,4$ ($\times$) (left) are localized close to the peaks of the quantum return probability R^T (right) obtained for the sloppy baker map with $\Delta=1/4$ and $N=96$.

After 30 iterations of the classical map the density distribution is close to the invariant measure f^*. Also the quantum state $\mathbf{B}_\Delta^{30}(|\alpha_0\rangle\langle\alpha_0|)$ is close to the invariant density matrix

$$\rho^*=\mathbf{B}_\Delta(\rho^*), \qquad (15)$$

the existence of which is guaranteed by the trace preserving condition (5). The state ρ^* is localized on the rectangle $[0,1]\times[0,1-\Delta]$. Moreover, it is almost isotropic on the corresponding $[N(1-\Delta)]$-dimensional subspace. To show this, we verified that the von Neumann entropy of the invariant state $S(\rho^*)=-\mathrm{Tr}\,\rho^*\ln\rho^*$ is close to the maximal entropy for the $[N(1-\Delta)]$-dimensional subspace of the Hilbert space $\mathcal{H}_N$,

$$\mathrm{S}(\rho^*)\approx \mathrm{S}_{\max}^{[N(1-\Delta)]}:=\ln(N(1-\Delta)). \qquad (16)$$

It is instructive to look at the periodic orbits of the classical transformation Θ_Δ. They are those of the original (reversible) baker map with momentum scaled by the factor $(1-\Delta)$,

$$q_T^*=\frac{n}{2^T-1}, \quad p_T^*=\frac{r(n)}{2^T-1}(1-\Delta), \qquad (17)$$

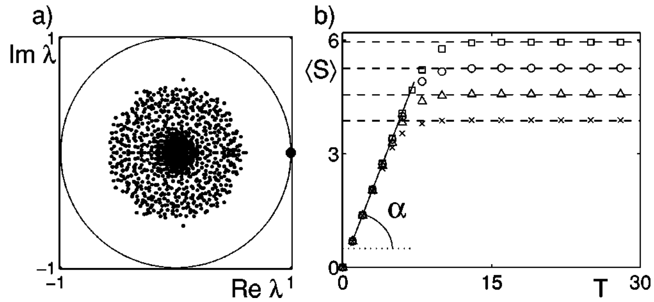

FIG. 4. (a) Eigenvalues of the superoperator $\mathbf{B}_\Delta$ for $N=64$ and $\Delta=1/4$ in the complex plane, larger dot denotes λ_1. (b) Dependence of the mean von Neumann entropy $\langle S\rangle$ on time T for the irreversible quantum baker map with $\Delta=1/4$ and $N=64(\times),128\,(\triangle),256\,(\bigcirc)$, and 512 ($\square$). Horizontal lines represent the asymptotic estimation (16).

where T denotes the length of the period, n ranges from 0 to (2^T-1), and the symbol $r(n)$ denotes the number obtained from n by reversing the order of its bits. The classical periodic orbits may be compared with the structures of the quantum return probability

$$R^T(q,p)=\langle q,p|\mathbf{B}_\Delta^T(|q,p\rangle\langle q,p|)|q,p\rangle. \tag{18}$$

The function $R^T(q,p)$ measures the projection of the quantum state $|q,p\rangle\langle q,p|$ iterated T times by the superoperator $\mathbf{B}_\Delta$ onto itself. As shown in Fig. 3, its maxima are indeed located in the vicinity of classical periodic orbits.

Spectral decomposition of the superoperator $\mathbf{B}_\Delta$ determines the time evolution of the system. For any trace preserving CP map (4) the operator $\mathbf{\Lambda}$ has an eigenvalue $\lambda_1=1$ corresponding to the invariant state ρ^*. The spectrum is symmetric with respect to the real axis, since $\mathbf{\Lambda}$ sends the Hermitian density matrices into density matrices [15]. Not every superoperator needs to be diagonalizable, i.e., the number of eigenvectors may be smaller than the number of eigenvalues. This is the case for the superoperator of translation $\mathbf{\Lambda}_\Delta$, the spectrum of which consists of two eigenvalues 0 and 1. The multiplicity of the former is equal to $3N^2/4$, and the corresponding subspace is defective for any $\Delta>0$.

Figure 4(a) shows all N^2 eigenvalues of the linear operator $\mathbf{B}_\Delta$ for $N=64$ and $\Delta=1/4$. Observe a considerable *spectral gap*, i.e., the difference $1-|\lambda_2|$, which determines the rate of the convergence of an initial state toward the invariant density matrix ρ^*. Moduli of the largest subleading eigenvalues influence the slope α of the initially linear entropy increase, directly related to the quantum dynamical entropy of the system (see, e.g., [16]). The data shown in Fig. 4(b) were obtained by averaging over a sample of ten initially pure states drawn randomly with respect to the unique, unitarily invariant measure on the $(2N-2)$-dimensional space of pure states in $\mathcal{H}_N$.

In this work we introduced an irreversible baker map and proposed a method of its quantization. On one hand, the limit $N\rightarrow\infty$ of the model may be useful to analyze the quantum-classical correspondence for chaotic, completely positive quantum maps. On the other hand, the extreme quantum regime of low N may be interesting from the point of view of quantum information. Quantum baker map becomes a standard model for theoretical [17] and experimental [18] investigations of nuclear magnetic resonance quantum computing, and our generalization makes it possible to study the consequences of irreversibility in the system. The effects of the decoherence and the dynamics of entanglement in the two-qubit version of this system $(N=4)$ as well as a generalization of the model will be presented elsewhere.

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